Frege and numbers as self-subsistent objects[*]

Frege y los números como objetos auto-subsistentes

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Resumen

En este artículo se argumenta que Frege no es el metafísico platónico sobre matemáticas que se considera normalmente. Se muestra que el proyecto fregeano tiene dos etapas distintas: la identificación de lo que es verdadero en nuestras nociones ordinarias, y luego la provisión de una explicación sistemática que comparte los aspectos identificados. Ninguna de estas etapas involucra mucha metafísica. El artículo critica en detalle la interpretación que hace Dummett de los parágrafos §§55-61 del *Grundlagen*. Estas secciones están bajo el encabezado 'Todo número es un objeto auto-subsistente' y Dummett las describe como las que contienen los peores argumentos planteados por Frege. Se arguye que, esencialmente, todos los puntos interpretativos de Dummett son erróneos. Finalmente, muestro que los planteamientos de Frege sobre la independencia de las matemáticas con respecto a los humanos y sus actividades tampoco lo comprometen con ninguna posición metafísica particular.

Palabras clave

Dummett, Frege, Grundlagen, independencia de las Matemáticas, platonismo metafísico.

Abstract

This paper argues that Frege is not the metaphysical platonist about mathematics that he is standardly taken to be. It is shown that Frege's project has two distinct stages: the identification of what is true of our ordinary notions, and then the provision of a systematic account that shares the identified features. Neither of these stages involves much metaphysics. The paper criticizes in detail Dummett's interpretation of §§55-61 of *Grundlagen*. These sections fall under the heading 'Every number is a self-subsistent object' and are described by Dummett as containing the worst arguments put forward by Frege. It is argued that essentially all of Dummett's interpretive points are mistaken. Finally, I show that Frege's claims about the independence of mathematics from humans and their activities does not commit him to any particularly metaphysical position either.

Key words

Dummett, Frege, Grundlagen, independence of Mathematics, metaphysical platonism.

INTRODUCTION

Frege is often described as perhaps the most prototypical realist in the philosophy of mathematics. It is important, however, to clearly separate what Frege says in terms of identifying desiderata for a systematic treatment of number from his remarks about how such a systematic treatment is to proceed. I contend that when this is done, what emerges is a picture of Frege as someone quite unconcerned with *metaphysical* issues. Neither of the two stages in providing a systematic account of arithmetic is particularly metaphysical. I hope mto show that Frege is largely unconcerned with establishing metaphysical theses of any kind.

The present paper has three sections. In the first I give a definition of metaphysical realism that ties the question of whether one is engaged in metaphysical issues to the aims of the project. I look at Frege's statement of his aims and show that they do not support a metaphysical reading of his program. In the second section, I analyze in detail Dummett's remarks (Dummett, 1991) on Frege's arguments in §§55-61 of *Grundlagen.* I argue that Dummett's interpretation is implausible, and implausible precisely because he interprets Frege as a metaphysical realist. In the third section, I turn to the question of Frege's remarks about the independent existence of mathematical objects and the independent truth of mathematical propositions. I argue that, here as well, there is nothing about these remarks that is particularly *metaphysical*.

1. FREGE'S AIMS AND METAPHYSICAL REALISM

To properly understand Frege's attitude towards the question of realism we need to look first at his goals. By carefully examining Frege's stated aims, it becomes apparent that he is not the metaphysical realist that he is standardly taken to be. In fact, I believe that the connection between one's overall goals and one's stance on realism is so strong that one can be defined in terms of the other. That is, I will take it as definitional of the position of *metaphysical realism* that anyone who holds this position has as their overarching goal to describe how reality is in some ultimate sense[1]. In defining *metaphysical realism* in this way, I do not take myself

[1] Erich Reck, in his *Frege's infuence on Wittgenstein: reversing metaphysics versus the context principle.* (In: TAIT, W. W. (Ed.). Early Analytic Philosophy: Frege, Russell, Wittgenstein. (pp. 123-85). Chicago and La Salle, IL: Open Court, 1997), argues that Frege is not a metaphysical realist in an argument that likens Frege's use of the context principle to the philosophy of the later Wittgenstein.

to be diverging from its standard sense. One is a metaphysical realist if, with your account –or of number, say– you try to correctly picture how the world is in some ultimate sense. To take a particularly fitting example, if one is giving an account that holds that numbers are objects, and in fact extensions of a certain kind, then that account is acceptable only if numbers *really are* particular kinds of extensions[2]. So, if being a metaphysical realist is a question of what one's ultimate goals are, then examining what one says about one's goals can help determine if one is a realist. Let us now, then, examine Frege's remarks concerning the goals of his project:

In the introduction to *Grundlagen*, Frege describes his project by saying: "I realize that, as a result, I have been led to pursue arguments more philosophical than many mathematicians may approve; but any thorough investigation of the concept of number is bound always to turn out rather philosophical. It is a task that is common to mathematics and philosophy"[3].

We see here Frege saying that he was forced by the nature of the subject matter to be somewhat philosophical. So, this already calls into question any view that holds that *from the outset* Frege sought to defend a position on the *metaphysical* status of arithmetic. He continues describing his project by saying: "Even I agree that definitions prove their worth by being fruitful"[4]. Here 'Even I' is in reference to Frege's standards of proof, and is not a comment on his realism. Again: "Definitions show their worth by proving fruitful. (...) Let us try, therefore, [to see] whether we can derive from our definition of the Number which belongs to the concept F any of the well known properties of number"[5] or again consider: "Now our concern here is to arrive at *a concept of number usable for the purpose of science*; we should not therefore, be deterred by the fact that in the language of everyday life number appears also in attributive constructions. That can always be got round"[6].

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[2] Being a metaphysical realist in this sense is closely related to holding a correspondence theory of truth. See RICKETTS, T. "Objectivity and objecthood: Frege's metaphysics of judgement". In: HAAPARANTA, L. & HINTIKKA, J. (Eds.). Frege Synthesized. (pp. 65-95). Dordrecht: Reidel, 1986; "Frege on logic and truth". In: Proceedings of the Aristotelian Society Supplement, 1996. vol. 70, p. 121-140 and RECK, E. H. "Frege on thruth, judgement, and objectivity". In: Grazer Philosophische Studien, 2007. vol. 75, no. 1, p. 149-173, for convincing arguments that Frege did not see truth in this way.

[3] FREGE, G. The Foundations of Arithmetic. 2d. ed. Evanston, IL: Northwstern University Press, 1980. p. v.

[4] *Ibid*., p. ix.

[5] *Ibid*., §70.

[6] *Ibid*., §57, my emphasis.

What, then, can we say about Frege's goals? Frege believes that the goal in providing an analysis of number is to put forward a definition that is "usable for the purpose of science". This condition is met if, from the definition, the ordinary properties of number are recoverable. So here we have what we can call Frege's first desideratum concerning a definition of number. Let us call *this the desideratum of standardness*.

Asecond desideratum becomes apparent when Frege provides his own definition of numbers as extensions of second-order concepts. He states "That this definition is correct will perhaps be hardly evident at first. For do we not think of the extensions of concepts as something quite different from numbers? (...) [I]t is not usual to speak of a Number as wider or less wide than the extension of a concept; but neither is there anything to prevent us from speaking in this way, if such a case should ever occur"[7].

Here we see Frege expressing some tolerance toward a definition of number. The definition need not match up exactly with our ordinary notion. The definition might confer on numbers properties that they did not previously have. Frege believes that the new properties conferred on number by his definition are harmless in the sense that they will arise so seldom as to not significantly affect the practice of arithmetic. This then can be seen as Frege's second desideratum for an account of arithmetic. That is, an account should not assign new properties to number in such a way that it will have a serious impact on the practice of mathematics. Let us call this desideratum *the desideratum of harmlessness.*

We have now identified Frege's two desiderata for a definition of number. A definition should allow us to recover the properties we usually associate with number. It should also not introduce new properties that would alter the practice of mathematics. When he is criticizing competing definitions, it is clear that he is applying the same standards. Consider one of his earliest criticism of a rival view that is presented in the *Grundlagen*: "When Stricker, for instance, calls our ideas of number motor phenomena and makes them dependent on muscular sensations, no mathematician can recognize his numbers in such stuff or knows what to make of such propositions"[8].

[7] *Ibid.*, §69.
[8] *Ibid.*, p. v.

The problem with Stricker's definition is that number as he defines it has essentially nothing in common with number as we ordinarily conceive it. Again consider Frege on a psychologistic accounts of number:

> If the number two were an idea, then it would have straight away to be private to me only. Another man's idea is, *ex vi termini*, another idea. We should then have it might be many millions of twos on our hands. We should have to speak of my two and your two, of one two and of all twos. If we accept unconscious ideas, we should have unconscious twos among them, which would return subsequently to consciousness. As new generations of children grew up, new generations of twos would continually be born, and in the course of millennia these might evolve, for all we could tell, to such a pitch that two of them should make five. Yet in spite of all of this, it would still be doubtful whether there existed infinitely many numbers, *as we ordinarily suppose*. 10^{10}, perhaps, might be only an empty symbol, and there might exist no idea at all, in any being whatever, to answer to that name[9].

One might interpret this as a straightforward *reduction ad absurdum*. On such an interpretation Frege's goal here is to show that *it is false* that numbers are ideas by showing that it leads to absurdity. However, there is a problem with this interpretation. If this were his goal here, then when he presents his own definition of numbers as extensions, he would need to provide an argument to the effect that *this is true*. Of course, he provides no such argument. In fact, he says that although it might be somewhat artificial to take numbers to be extensions, there is nothing wrong with this. In light of the two identified desiderata, another interpretation becomes evident. Frege is not concerned with the question of whether numbers *really are ideas or not*. Frege is rejecting attempts to define numbers psychologistically precisely because such a definition would satisfy neither the desideratum of standardness, nor the desideratum of harmlessness. In fact the failure of one or the other of these desiderata is argued for in each sentence in the above quote.

Does what I have said so far show that Frege is not a realist? That is not what I want to claim at all. Frege takes numbers to be objects and takes mathematical claims to express objective truths. So, he is clearly a realist in a certain sense. What I am claiming is that there is nothing

[9] *Ibid.*, §27, my emphasis.

metaphysical about his realism. When Frege claims that numbers are objects and mathematical claims express objective truths, he is not making a metaphysical pronouncement. What he is doing is stating which features an account of number must have in order to satisfy what I have called his desideratum of standardness. Frege takes these to be properties of numbers as we ordinarily conceive of them, and therefore properties that ought to be preserved by an account of number. He does not take this to make any metaphysical claim about the ultimate nature of reality. In fact he warns against reading his claim that numbers are self-subsistent objects in this way: "The self-subsistence which I am claiming for number is not to be taken to mean that a number word signifies something when removed from the context of a proposition, but only to preclude the use of such words as predicates or attributes, which appreciably alters their meaning"[10].

In this section I have tied the question of the degree to which an account of arithmetic is realist in a metaphysical sense to the expressed goals of the account. If the goal of the account is to reflect the ultimate nature of reality (or at least those features that concern the subject matter of the account), then it is a metaphysically realist position. We have seen that Frege's stated goals are actually quite pragmatic. Frege wants to give an account of arithmetic "suitable for the purposes of science". Although he claims, for instance, that numbers are self-subsistent objects, he does not mean for this to correctly picture how things in reality actually are in any metaphysical sense. As Ricketts (1986), Ricketts (1996), and Reck (2007), for instance, argue, Frege does not accept anything like the correspondence theory of truth.

Even if, however, Frege's ultimate goal was not to give a theory that corresponds to the way the world is in itself, the question of truth cannot be completely avoided. Truth is a concept of central importance for Frege. In a language like that of *Begriffsschrift*, all of the provable judgements express truths. How is the quite pragmatic reading of Frege defended here consistent with the claim that provable judgements in Frege's systematic languages express truths? To answer this we need to explore Frege's discussion of the paradox of analysis. The paradox of analysis states that no analysis can be both informative and correct. If we say what it is to be *A* is to be *B*, then there are two possibilities. If '*B*' has exactly the same meaning as '*A*', then the analysis is uninformative.

[10] *Ibid.*, §60.

If '*B*' means something different, then the account is incorrect. Frege provided several attempts at a solution to the paradox of analysis, but I will, here, focus on his final solution to the paradox[11]. Frege thought that even the pre-systematic use of scientific terms, presumably including 'number', lack a completely clear meaning: "Now it may happen that this sign (word) is not altogether new, but has already been used in ordinary discourse or in a scientific treatment that precedes the truly systematic one. As a rule, this usage is too vacillating for pure science"[12].

If even the scientific treatment of concepts such as 'number' has a use that is too vacillating, how is an analysis supposed to work? The answer is provided in 'Logic in Mathematics':

> If we have managed in this way to construct a system of mathematics without the need for the sign *A* [a sign with an existing sense], we can leave the matter there; there is no need at all to answer the question concerning the sense in which –whatever it may be– this sign had been used earlier. In this way we court no objections. However it may be felt expedient to use the sign A instead of the sign B [the newly introduced sign]. But if we do this, we must treat it as an entirely new sign which has no sense prior to the definition. We must explain that the sense in which this sign was used before the new system was constructed is no longer of any concern to us, that its sense is to be understood purely from the constructive definitions that we have given. In constructing the new system we take no account, logically speaking, of anything in mathematics that existed prior to the new system. Everything has to be made anew from the ground up. Even anything we accomplish by our analytical activities is to be regarded only as preparatory work which does not itself make any appearance in the system itself[13].

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[11] Frege ("Review: Husserl, philosophy of arithmetic". In: McGUINNESS, B. (Ed.). Collected Papers on Mathematics, Logic and Philosophy (pp. 293–340). Oxford: Basil Blackwell, 1984a) discusses the paradox explicitly (although, of course, not by that name). Michael Beaney discusses Frege's views on analysis in BEANEY, M. Frege: Making Sense. London: Duckworth, 1996. chap 5 and his "Carnap's conception of explication: from Frege to Husserl?" (In: AWODEY, S. & KLEIN, C. (Eds.). Carnap Brought Home: The View from Jena (pp.117–150). Chicago and La Salle, IL: Open Court, 2004) including looking at his earlier attempts to avoid the paradox. Joan Weiner discusses problems related to Fregean analyses in WEINER, J. "What's in a numeral? Frege's answer". In: Mind, 2007. vol. 116, no. 463, p. 677-71).

[12] FREGE, G. "On the foundations of geometry: second series". In: McGUINNESS, B. (Ed.) Collected Papers on Mathematics, Logic and Philosophy (pp. 293–340). Oxford: Basil Blackwell, 1984b. p. 302-302.

[13] FREGE, G. "Logic in mathematics". In: HANS HERMES, F. K. & KAMBARTEL, F. (Ed.). Poshumous Writings (pp. 203-250). Oxford: Basil Blackwell, 1979. p. 211.

We see here Frege's final view on how an analysis is supposed to function. We begin by identifying what is true of the notion prior to its truly systematic treatment. This, however, is only the first step, and is a somewhat vague matter. We then replace the old term with a new one (or retain the old one *as the new one* for convenience). The new term's meaning, however, is to follow exclusively from the definition –ties to the old notion are severed. As I have explained Frege's aim in *Grundlagen*, he is engaged very much in a project along these lines. He does not argue that numbers really are and always have been extensions. He offers his definition as a replacement for our ordinary notion of number.

At this point, it will evidently be charged that I am reading Frege's remarks from 'Logic in mathematics' back into *Grundlagen*, despite the two works being separated by thirty years. Of course, Frege did not have his views on analyses expressed in 'Logic in mathematics' worked out at the time of *Grundlagen*. This is evidenced by his formulating several intermediate positions on analysis in the intervening years. However, even if Frege did not have *his general views* on explication worked out by the time of *Grundlagen*, what he does in *Grundlagen* is nonetheless *an instance of* such an analysis. Frege makes no attempt to argue that numbers always were extension. He thinks that number as he defines it is sufficiently similar to the ordinary notion that it could be used in its place. In this section I have argued for an interpretation of Frege where his aims are more practical and less philosophical than standardly assumed. In the next section I turn to an interpretation at odds with this one.

2. DUMMETT ON §§55-61

Michael Dummett has defended a strongly realist interpretation of Frege. In fact, Dummett's interpretation amounts to an interpretation of Frege as metaphysical realist in exactly the sense defined in the last section. Dummett takes Frege to hold that an account of arithmetic ought to reflect how things are in the world –and in particular the ontology of the world. Dummett writes "Frege's realism was not the most important ingredient in his philosophy: but the attempt to interpret him otherwise than as a realist leads only to misunderstanding and confusion"[14] I do not want to deny that Frege was a realist. Frege believed that numbers

[14] DUMMETT, M. "Frege as a realist". In: SLUGA, H. (Ed.). Meaning and Ontology in Frege's Philosophy, Vol. 3 of The Philosophy of Frege (pp. 109-122). New York & London: Garland, 1993. p. 109.

are objects, and that numerical statements express objective truths. But unlike Dummett I don't see these as Frege's conclusions. Rather, these express desiderata concerning an account of number. So Frege certainly was a realist –just not a metaphysical realist. The measure of an account is not whether it correctly reflects the true structure of metaphysical reality, but that it preserve the properties that we standardly take numbers to have. In this section I wish to show that my interpretation of Frege's realism allows for a clearly more appropriate interpretation of §§55-61. Dummett (1991) describes §§55-61 of *Foundations* (the sections that fall under the heading 'Numbers are self-subsistent objects') as containing some of the worst philosophical arguments presented by Frege[15].

Dummett discusses these sections at length in the chapter of *Frege: Philosophy of Mathematics* titled 'Two strategies of analysis'. Here, Dummett distinguishes between two strategies for defining number. On the one hand, we could be radical substantivalists and begin by defining numerical terms and then translate adjectival uses of number into claims involving numerical terms. On the other hand, we can define adjectival uses of number and then translate all statements which involve numerical terms into ones that do not. This second strategy Dummett calls the *radical adjectival strategy*. In §55, as Dummett understands him, Frege proposes three definitions that pursue a radical adjectival strategy. That is, the definitions attempt to define our use of numbers as adjectives without the use of numerical terms. The definitions are stated as follows:

> the number 0 belongs to a concept, if the proposition that a does not fall under that concept is true universally, whatever a may be.
>
> the number 1 belongs to a concept F, if the proposition that a does not fall under F is not true universally, whatever a may be, and if from the propositions that "a falls under F" and "b falls under F" it follows universally that a and b are the same.

[15] Wray, K. B. focusses exclusively on Dummett's interpretation of x56. Although I agree with Wray that Dummett seriously misreads Frege here, I don't see that Wray has identified what exactly it is about Dummett's interpretation that is at fault. I do not think it is simply that "Frege is not trying to show that the adjectival strategy cannot make sense of our use of number statements, as Dummett suggests, but rather, his intention is to show that only the substantival strategy can provide us with an adequate definition of number" ("Reinterpreting section 56 of Frege's The Foundations of Arithmetic". In: Auslegung: A Journal of Philosophy, 1995. vol. 20, no. 2, p. 79). The main reason I don't think Wray's interpretation is acceptable is that it is not altogether clear how it differs from Dummett's. Presumably, if only a substantival view will be acceptable, then this itself is a strong argument against the adjectivalist.

> the number (n+1) belongs to the concept F, if there is an object a falling under F and such that the number n belongs to the concept "Falling under F, but not a"

However, on Dummett's view:

> In stating them, [Frege] makes heavy use of his jargon. Instead of saying 'There is just 1 F', (…) he says 'The number 1 belongs to the concept F'. This, of course, obscures the fact that these are *adjectival* uses of number words that he is defining[16].

In §56 Dummett sees Frege as objecting to the adjectival definition of number just presented in §55. Frege presents two arguments here. The first involves the use of the variable n in the third definition. If numerical terms have not been defined, but only phrases containing numerical terms, then we are not free to use a variable that ranges over numbers. Dummett recognizes this as a valid criticism of Frege's proposed definition. Dummett argues, however, that if Frege had taken the adjectival strategy more seriously he would have realized that this objection is not valid against all attempts to give an adjectival definition of number. As a result, Dummett spends much of the chapter providing an adjectival definition that avoids this difficulty while at the same time involving no numerical terms. Frege mentions, as an illustration of this problem with the third definition, that the proposed definition does not rule out Julius Caesar as a number. Dummett can make no sense of this; he merely states that no one reading the work for the first time could see this as relevant, and then he moves on.

The second objection that Frege puts forward is that the definitions do not allow us to show that if *a* and *b* are both numbers belonging to a concept *F*, then *a* = *b*. Concerning this, of course, Dummett points out that identity, for Frege, is a relation between objects. As such, Dummett sees this argument as completely question begging. As with his handling of the first objection, Dummett shows that with proper definitions this problem can be avoided. He does this by defining a relation between numerically definite quantifiers that can be used to show that only one numerically definite quantifier applies to a given concept. Concerning §56, Dummett writes: "Frege aimed to show all

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[16] DUMMETT, M. Frege: Philosophy of Mathematics. Cambridge, MA: Harvard University Press, 1991. p. 100.

three definitions erroneous, and thereby to prove a purely adjectival strategy unfeasible, because numbers have to be recognized as being objects. In this he utterly failed: in fact §56 may be stigmatized as the weakest in the whole of Grundlagen. The arguments lack all cogency: they more resemble sleight of hand"[17].

Dummett sees §57 as merely a continuation of §56. Therefore, he sees it as just as question begging as the previous section. In addition to this, Dummett finds Frege to be self-undermining. Here Frege shows how we can translate adjectival uses of number into statements involving singular terms standing for numbers. But if we are free to reinterpret in one direction, why should we take the surface syntax seriously in the other? Dummett concludes that Frege has given no good argument that numbers are objects. Concerning §§58-61, Dummett has no objections. He merely points out that these sections contain quite admirable defense of the claim that numbers are objects against the possible objections that we can have no ideas of them or that they have no spatial location.

I seriously disagree with almost every aspect of Dummett's interpretation of §§55-57. In the previous section, I argued that Frege is not a metaphysical realist. The way that Dummett interprets §§55-57 is possible only on the assumption that Frege is a metaphysical realist. Dummett thinks that Frege is concerned to show that numbers really are objects, and sees §§55-57 as a sustained, but incredibly poor, argument to this effect. Dummett is here reading far too much into the text.

Dummett interprets Frege as proposing for consideration radical adjectival definitions of number in §55, and in so doing, using incredibly questionbegging jargon. However, there is nothing about §55 which suggests that Frege is considering a radical adjectival definition of number. After presenting his 'fundamental thought' in §46 and discussing it in the following sections, Frege says that this leads almost immediately to the definitions of §55[18]. In all three of these definitions, Frege includes what are clearly meant to be terms that stand for numbers (as objects). If one interprets these definitions as following the radical adjectival strategy, as Dummett does, the use of the definite articles that

[17] *Ibid.*, p. 105.

[18] On page 5 of Frege *The Basic Laws of Arithmetic: Exposition of the System* (Edited and translated by Montgomery Furth. Berkeley and Los Angeles: University of California Press, 1967), Frege describes the claim that statements of numbers contain an assertion about a concept as the fundamental result of *Grundlagen*.

appears in each of the three definitions must be chalked up to either *a major oversight or major dishonesty*. It is not clear which option is meant to be implied by Dummett's phrase 'question begging jargon'. If on the other hand, we assume that Frege chose consciously and without dishonesty to use the definite article in the definitions, then §55 contains a discussion of readily suggested but ultimately unacceptable definitions *of numerical terms*. Frege understands 'the number n belongs to F' as having the form *B* (*n, F*), where *B* (the 'belongs to' relation) is a relation between an object and a first level property. On this interpretation, §55 does not contain radical adjectival definitions *at all*. Instead, what Frege is considering here is introducing numerical terms via the role they play in numerically definite quantifiers.

What then of §56? Of course, since §55 does not contain a radical adjectival definition of number, §56, which discusses these definitions, does not consider a radical adjectival strategy either. What Frege considers in this section is whether the numerical terms have been sufficiently well-defined. He concludes that they have not. The first reason given involves the introduction of the variable n ranging over the numbers in the third definition. His objection to this is that the variable ranging over numbers is illegitimate since the numbers have not yet actually been defined. To support this last point Frege mentions the Julius Caesar problem. Dummett, as we saw, says little of the mention of Caesar here other than that it is baffling. This is not surprising. If the definitions of §55 are not meant to contain numerical *terms*, then the substitution of Julius Caesar for a number *does not make any sense at all*. But on an interpretation where the definitions in §55 are meant to define numerical terms –that is terms for numbers as objects– it makes perfect sense. If numbers are objects, then a definition of number should tell you which objects are numbers and which are not. Dummett may be right that the mention of Caesar may be completely unexpected by first time readers, but as surprising as it maybe, on the interpretation defended here it is perfectly well motivated.

Frege then points out that it also does not follow from the definitions that if *a* and *b* are both numbers that belong to the same concept, that *a* = *b*. Dummett, in holding that the radical adjectival strategy is under consideration, sees Frege, by considering identities, as once again 'taken in by his own jargon' and assuming what he is trying to prove. However, if the definitions in §55 were never meant to be radical adjectival definitions, then a far more charitable interpretation is possible. If Frege intended the definitions in §55 as definitions of numerical terms, then Frege is merely pointing out that the definitions do not allow us to derive a basic property of numbers –that for any (sortal) concept there is one and only one number of things that falls under it. Remember, we saw in the previous section that deriving the basic properties of number was Frege's criterion of success for a definition of number.

Section 56 closes with the complaint that the definitions have only fixed the sense of the phrases 'the number 0 belongs to', 'the number 1 belongs to' but they do not allow us to pick out 0 and 1 as self-subsistent objects. Dummett sees §56 as intending to address the radical adjectivalist, and sees this remark as again incomprehensibly question begging. However, if the definitions were intended to introduce numerical terms, as the use of the definite article clearly suggests, then there is nothing question begging here. Frege is simply pointing out that the phrase 'the number 1' has not been defined other than as part of 'the number 1 belongs to'. That is, although the definition makes clear the contribution of the phrase 'the number 1 belongs to' to sentences containing it, it does not show what the contribution of 'the number 1' is. Since, on the interpretation defended here, Frege is attempting in §55 to define numerical terms, this points to what is obviously a significant flaw in the definitions.

In §57 Frege does finally consider adjectival uses of number. This section does not merely continue the attack on the radical adjectival definition of number, as Dummett insists, but is in fact, the first place where purely adjectival uses of number are considered. Frege begins, however, by discussing the fundamental thought –"the content of a statement of number is an assertion about a concept"[19]. Frege is concerned that this might lead one to conclude that numbers are properties of concepts. If we examine statements of the form 'the number *n* belongs to the concept *F*', *n* appears as only part of what is predicated of *F*. As a mere part of the predicate we are not obliged to view numbers as properties of concepts. Frege then says "Precisely because it forms only an element of what is asserted, the individual number shows itself for what it is, a self subsistent object". This may sound question begging, but it is not. The justification for this is not, as Dummett thinks, what was said in §56, but what comes immediately after. After this remark Frege says why he seeks to define numbers as objects. His first reason is that we often

[19] FREGE, The Foundations of Arithmetic. Op. cit., §46.

attach the definite article to numbers. The second reason is that Frege believes that equations are best viewed as identities[20].

Now, if, on the one hand, Frege's goal were to show that numbers *really are* objects, then the reasons given in §57 are incredibly flimsy.[21] I imagine it is for this reason that Dummett looks for further arguments against the radical adjectivalist in §56. Since the two, briefly stated, reasons are *clearly* insufficient to prove a strong metaphysical thesis, Dummett reads arguments for this thesis into the preceding sections. However, if, on the other hand, Frege's goal were to "arrive at a concept of number usable for the purpose of science" –as he explicitly states it is *at this very point in the text!*–, then these reasons may be sufficient to motivate a definition of numbers as objects.

It might be objected that if §§55-56 do not consider the radical adjectival definition of number, then they ought not appear under the general heading 'Every number is a self-subsistent object' as the sections from 55-61, in fact, do. We are now in a position to answer this objection. We saw that the definitions in §55 were put forward as being strongly suggested by the fundamental thought. These definitions were never meant to follow the adjectival strategy, but merely to pursue what might be suggested by the fundamental thought. Frege wants to show that even in these definitions, numbers are not properties of concepts. The inclusion of §§55-56 under this heading is to show that the fundamental thought does not force us to view numbers as properties of concepts. If Frege is right that the definitions given in §55 'suggest themselves so spontaneously in light of our previous results [the fundamental thought]', then showing that they don't commit us to a view of numbers

[20] Frege does give an argument for why we should view equations as identities in (FREGE, Logic in mathematics, Op. cit., p. 223). Here Frege argues against interpreting an equation as a problem and a solution. If we interpret the left hand side of , say, '4 + 3 = 7' as a problem and the right hand side as a solution, then what are we to make of '(4 + 3) + 2 = 9'? It seems the left hand side would then be a problem of adding 2 to a problem. This argument may not rule out viewing equations as identities between entities of types other than objets, but shows that his interpretation of them as identities avoids a problem with an alternative view.

[21] William Demopoulos ("Critical notice of Michael Dummett's 'Frege". In: Philosophy of mathematics' Canadian Journal of Philosophy, 1993. vol. 23, no. 3, p. 477-496) argues that there is a further kind of justification for Frege's taking numbers to be objects. Taking numbers to be objects allows Frege to prove that there are infinitely many numbers. Demopoulos compares this to the derivation of Kepler's laws from Newton's theory. In the second case it would be absurd to claim that the derivation of Kepler's laws was merely a motivation for Newton's theory, and that it offers no justificatory support. I admit that the derivation of the well known properties of numbers does offer some kind of support for the claim that numbers are objects. But exactly how it lends support and whether or not it lends any support to a metaphysical realist's interpretation of the claim that numbers are objects, I will not settle here.

as properties of concepts is important, since Frege intends to define numbers as objects. Dummett eventually puts forward an adjectival definition of number that avoids the problems that Frege identifies with his own definition, but does not defend the claim that these definitions are suggested immediately by the fundamental thought. I will not myself weigh in on whether Frege is right that it is the definitions in the form he provides which most naturally suggest themselves in light of the fundamental thought. But I will simply point out that *if he is right*, then the fundamental thought does not give us any reason to treat numbers as anything other than objects. This is because the definitions contain, explicitly and as I have argued intentionally, numerical terms.

Frege does eventually, still in §57, consider constructions such as 'Jupiter has four moons'. In these constructions, 'four' is not a mere part of a predicate. Here, for the first time in these sections, a straightforwardly adjectival construction is under consideration. But we see that Frege makes no attempt to show that interpreting numbers as objects gets things right in some ultimate metaphysical sense, and that interpreting numbers adjectivally would get things wrong. His attitude at this point is purely pragmatic. He merely claims (as quoted earlier): "we should not (…) be deterred by the fact that in the language of everyday life number appears also in attributive constructions. That can always be got round"[22].

Remember that for the present purposes, 'metaphysical realism' was defined in terms of the criteria one accepts for acceptability of the account one is putting forward. That is, does one take an important criteria to be that the account reflects how things actually stand in the world? The pragmatic attitude expressed here is incompatible with metaphysical realism. If there really is a fact of the matter of whether numbers are objects, and the measure of an account is that it reflects how things really are in the world, then this is a serious question that would deserve serious attention. Frege essentially devotes no attention to this question. All he does is give a couple of reasons for why he takes numbers to be objects. If one assigns Frege the lofty goal of showing that numbers are objects in the sense of the metaphysical realist, then Frege's arguments must be seen as unbelievably weak. However, if one interprets Frege as having the more modest goal of providing a firm foundation for mathematics, such that numbers have the various

[22] FREGE, The Foundations of Arithmetic, Op. cit.

features we standardly take them to have, then one can provide a far more palatable interpretation of these sections.

3. INDEPENDENCE

I have given an interpretation of Frege such that he is not defending any metaphysical position. When he claims that numbers are self-subsistent objects or that arithmetical claims state objective truths, he is merely stating what he takes to be true of our ordinary conception of number. In stating the properties of our ordinary notion of number, Frege is not engaging in metaphysics, but identifying what features a systematic treatment of number should preserve. When he does offer a systematic account, he takes it to be justified in terms of its agreement with ordinary notions: "The reader will recognize that my basic principles at no point lead to consequences that he is not himself forced to acknowledge as correct"[23].

Consider also this passage comparing psychological versus logical foundations:

> It is *prima facie* improbable that such a structure could be erected on a base that is uncertain or defective. Anyone who holds other convictions has only to try to erect a similar structure upon them, and I think he will perceive that it does not work, or at least does not work so well. As a refutation in this I can only recognize someone's actually demonstrating either that a better, more durable edifice can be erected upon other fundamental convictions, or else that my principles lead to manifestly false conclusions[24].

What is important about this quote, for our purposes, is what he says about what he would take as a refutation of his system. So long as it does not lead to false consequences or turns out to be demonstrably inferior to another system, it should be accepted. This hardly seems like the standards of someone who thinks the primary goal of an account is to correctly reflect how things are in the world in some ultimate sense.

[23] FREGE, The Basic Laws of Arithmetic: Exposition of the System. Op. cit., p. 9.
[24] *Ibid.*, p. 25.

But what of Frege's insistence that the truth of mathematical statements is independent of anything any human does? It may seem that this independence is in conflict with the interpretation provided here, but that is not the case. We will examine this on two fronts. First, we will look at the question of the independent existence of mathematical objects. We will then turn to the question of the of the independence of mathematical propositions.

Let us then begin with the question of the independence existence of mathematical objects. We saw above that when Frege identifies numbers with extensions, no effort is made to show that this is true in any absolute sense. He does not hold that numbers really are and have always been extensions, but thinks that such an identification in a systematic treatment of number is justifiable in that it satisfies the two desiderata discussed in the first section. But what of other objects Frege wished to include in his systematic treatment of arithmetic? There are two of these: truth-values and courses-of-values. We see that in both of these cases Frege cites practical reasons for their introduction. Concerning courses-of-values he writes: "The introduction of courses-of-values of functions is a vital advance, thanks to which we gain far greater flexibility"[25]. And concerning the truth-values he claims: "How much simpler and sharper everything becomes by the introduction of truth-values, only detailed acquaintance with this book can show. These advantages alone put a great weight in the balance in favor of my own conception, which indeed may seem strange at first sight"[26].

These hardly amount to arguments that truth-values and courses-of-values are objects in some ultimate metaphysical sense.

Of course, Frege does claim, repeatedly, that we cannot create objects with definitions. In *Grundgesetze* Frege says that the mathematician can no more create an object by definition than can the geographer create a sea by giving a particular portion of the Earth's surface the name 'the Yellow Sea'. Is Frege here making a metaphysical claim that numbers exist in some absolute sense? No, Frege is not engaging in metaphysics here at all. This can be seen from the clearly *logical* point that he goes on to make:

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[25] *Ibid.*, p. 6.
[26] *Ibid.*, p. 7.

> Now suppose one defines, for instance, the number zero, by saying: it is something which yields one when added to one. In so doing one has defined a concept, by specifying what property an object must have in order to fall under the concept. But this property is not a property of the concept defined. People frequently seem to fancy that by the definition something has been created that yields one when added to one. A great delusion! The concept defined does not possess this property, nor is the definition any guarantee that the concept is realized –a matter requiring separate investigation. Only when we have proved that there exists at least and at most one object with the required property are we in a position to invest this object with the proper name "zero". To create zero is consequently impossible[27].

So, basic principles that introduce objects of certain kinds (e.g. courses-ofvalues) are to be accepted or not on practical grounds. They are to be accepted if their introduction gives us the kind of system that we are looking for –that is, in this case, one that could play the required role in the foundations of mathematics. Frege's point, about it being impossible to create an object with a definition, is a logical and not metaphysical point. It is aimed at those who mistakenly believe that by defining a concept we thereby define into existence an object of which it is true. In neither of these cases, then, are Frege's concerns metaphysical.

Let us then turn to the question of the independence of assertions involving number. Here again, Frege believes that it is a feature of our ordinary conception of number that ought to be preserved by a systematic account. There is nothing all that metaphysical about imagining a world without people. If a biologist claims that if there had been no humans the wooly mammoth would still exist, this is not a metaphysical claim. It does not seem any more metaphysical to claim that it is true according to our ordinary conception of number that, even in a world without people, if a certain area contains two pebbles and also two other pebbles, then there are at least four pebbles in that area. Now if the truth conditions for statements of number involved ideas in people's minds, then nothing would be true of number in a world with no people. So Frege takes it that a definition of number should not mention people or their ideas. In a passage clearly related to the one discussed above, concerning the Yellow Sea, Frege writes: "If we say that "the North Sea is 10,000 square miles in extent" then neither

[27] *Ibid.*, p. 11-12.

by "North Sea" nor by "10,000" do we refer to the state of or process in our minds: on the contrary, we assert something objective, which is independent of our ideas and everything of the sort"[28].

Again, it is evident here that his target is psychologistic accounts of number. He is saying that the claim "the North Sea is 10,000 square miles in extent" makes no reference to people and so should be true whether or not there are any people. This independence is again identified as a desideratum of a systematic account of number. This desideratum is obviously not satisfied by a psychologistic account of number. But Frege does not hold that our presystematic understanding of number already have perfectly clear truth conditions: "Since it is only in the context of a proposition that words have meaning, our problembecomes this: To define the sense of a proposition in which number words occur. That, obviously, leaves us still a wide choice"[29].

Our ordinary conception of number has the feature that the truth of such claims is independent of us and any of our activities. However, in giving a systematic treatment of number, we still have some freedom in how we define number. What we want of such a definition is that it satisfies our quite practical desiderata.

Thus we see, Frege's talk of independence does not commit him to metaphysical positions. Concerning basic principles that introduce entities, Frege sees their acceptance as being a pragmatic question. In terms of the independent truth of mathematical statements, this need not be interpreted as involving any metaphysics[30].

CONCLUSIONS

We saw that Frege describes the purpose of his project as arriving at 'a concept of number usable for the purpose of science'. To this end, two

[28] FREGE, The Foundations of Arithmetic. Op. cit., §26.
[29] *Ibid*., §62.
[30] Burge, T. (Truth Thougt Reason: essays on Frege. Oxford & New York: Oxford University Press, 2005) repeatedly defends the position that the pragmatic aspect that I discuss are present in Frege, but at the end of the day, Frege takes the pragmatic success of his system to be evidence for its metaphysical truth. Although, I cannot, at this point, discuss Burge's arguments in detail, I do not think he has provided sufficient argument for the claim that despite all the pragmatic elements in Frege's writings, he is nonetheless a metaphysical realist. Burge's main line of argument for his position is the independence just discussed. But if this need not be interpreted metaphysically, then I see little reason for a metaphysical interpretation of Frege's writings.

desiderata were identified: the desideratum of standardness and the desideratum of harmlessness. The desideratum of standardness holds that the properties we ordinarily associate with number should also hold of numbers in a systematic treatment. The desideratum of harmlessness holds that any new properties a definition confers on number should not significantly change the practice of mathematics. A systematic treatment that satisfies these desiderata would seem to fully satisfy Frege. There is no further requirement that the systematic treatment reflect how things are in the world in some ultimate sense.

We then looked in detail at §§55-61 of *Grundlagen*. Whereas Dummett sees Frege as addressing the radical adjectivalist in §55 and §56, we saw there was nothing in those sections to suggest that. In fact, it is quite clear that Frege intends in these sections to define numerical terms. It is a further advantage of my interpretation, unlike that of Dummett, that it can make sense of the Julius Caesar objection as it appears in these passages. Only if it is numerical terms being defined is Julius Caesar a legitimate substitution instance. Finally, in §57, when Frege does consider purely adjectival uses of number, he expresses a purely pragmatic attitude. This is not consistent with a *metaphysical view* of numbers as objects.

In the last sections we looked at Frege remarks concerning the independent existence of number and the independent truth of arithmetical propositions. It was shown that on neither of these issues need we interpret Frege as holding a metaphysical thesis. When he says we cannot create objects with our definition, he is making a logical, rather than metaphysical, point. When he claims that arithmetical propositions are true completely independently of us, he is saying that we ought not define number such that truths of number depend on truths about us.

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